Journal of Approximation Theory **112**, 226–234 (2001)
doi:10.1006/jath.2001.3589, available online at http://www.idealibrary.com on IDEAL®

Approximation by Dirichlet Series with Nonnegative Coefficients

Yunkang Liu[1]

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom
E-mail: Y.Liu@lboro.ac.uk

Communicated by Tamás Erdélyi

Received May 1, 2000; accepted in revised form April 9, 2001

The problem of approximating a given function by Dirichlet series with nonnegative coefficients is associated with the discrete spectral representation of the relaxation modulus in rheology. The main result of this paper is that if a function can be approximated arbitrarily closely by Dirichlet series with nonnegative coefficients in supremum norm or L_p-norm, $1 \leqslant p < \infty$, then it must be completely monotonic. © 2001 Academic Press

Key Words: Dirichlet series; completely monotonic function; Müntz theorems; relaxation modulus; rheology.

1. INTRODUCTION

The discrete spectral representation of the relaxation modulus in rheology is associated with the problem of approximating a given real function (relaxation modulus) f over $R_+ := [0, \infty)$ by Dirichlet series of the form

$$f_n(t) = a_0 + \sum_{k=1}^{n} a_k e^{-\alpha_k t}, \tag{1}$$

where $a_0, a_1, a_2, \ldots$ are nonnegative constants and $\alpha_1, \alpha_2, \ldots$ are distinct positive constants; both sets can depend on n. Typical assumptions on the function f include (Joseph [5, p. 542])

$$f(t) > 0, \qquad f'(t) < 0, \qquad f''(t) > 0, \qquad \text{and} \qquad \lim_{t \to \infty} f(t) = 0. \tag{2}$$

According to Joseph [5, p. 569], there is at present no discussion on whether a smooth function, satisfying all or part of the assumptions listed in (2) can be approximated arbitrarily closely by Dirichlet series of the form (1). It will be shown in this paper that the answer to this question is

[1] Current address: 34 Cedar Close, Burkhurst Hill, Essex IG9 6EJ, United Kingdom.

0021-9045/01 $35.00

no. More specifically, there are functions that satisfy (2) but cannot be approximated arbitrarily closely by Dirichlet series of the form (1) in the function space $C(R_+)$ with the supremum norm or in $L_p(0, \infty)$, $1 \leqslant p < \infty$, with the usual L_p-norm. Therefore, a more appropriate question would be what kind of functions can be approximated arbitrarily closely by Dirichlet series of the form (1) in $C(R_+)$ or in $L_p(0, \infty)$.

Let Δ be R_+ or a subset of R_+, S the function space $C(R_+)$ equipped with the supremum norm or $L_p(0, \infty)$ with the usual L_p-norm, $1 \leqslant p < \infty$, and denote by

$$\mathrm{Span}\{e^{-\alpha t} \mid \alpha \in \Delta\}$$

the set of all linear combinations of finite numbers of $e^{-\alpha t}$, $\alpha \in \Delta$, with real coefficients,

$$\mathrm{Pspan}\{e^{-\alpha t} \mid \alpha \in \Delta\}$$

the set of all linear combinations of finite numbers of $e^{-\alpha t}$, $\alpha \in \Delta$, with nonnegative coefficients,

$$\overline{\mathrm{Span}}\{e^{-\alpha t} \mid \alpha \in \Delta \mid S\}$$

the closure of $\mathrm{Span}\{e^{-\alpha t} \mid \alpha \in \Delta\}$ in S, and

$$\overline{\mathrm{Pspan}}\{e^{\alpha t} \mid \alpha \in \Delta \mid S\}$$

the closure of $\mathrm{Pspan}\{e^{-\alpha t} \mid \alpha \in \Delta\}$ in S. The main objective of this paper is to give a good characterisation of $\overline{\mathrm{Pspan}}\{e^{-\alpha t} \mid \Delta \mid S\}$ for some Δ and S.

Section 2 of this paper deals with the case where $\Delta = R_+$ and $S = C(R_+)$. It will be shown that a function is in $\overline{\mathrm{Pspan}}\{e^{-\alpha t} \mid \alpha \in \Delta \mid S\}$ if and only if it is completely monotonic in R_+. Section 3 is concerned with the case where $\Delta = R_+$ and $S = L_p(0, \infty)$, $1 \leqslant p < \infty$. Results in these two section show that there are functions that satisfy (2) but cannot be approximated arbitrarily closely by Dirichlet series of the form (1) in $C(R_+)$ or in $L_p(0, \infty)$. In Section 4, we discuss the case where Δ is everywhere non-dense in R_+. This case has direct application in Rheology (see, e.g., [10]) where (1) is supposed to be the corresponding relaxation modulus of a generalized Maxwell model, i.e., $\alpha_k = 1/\lambda_k$, $k = 1, 2, \ldots$, with λ_k being the discrete relaxation times. The *Doi and Edwards molecular theory for polymer melts* [4, p. 228] suggests

$$\lambda_k = \lambda_0/(2k+1)^2, \qquad k = 1, 2, \ldots \tag{3}$$

while the *Rouse molecular theory for dilute polymer solutions* (see, e.g., Doi and Edwards [4, p. 226]) gives

$$\lambda_k = \lambda_0 / k^2, \qquad k = 1, 2, \ldots \tag{4}$$

with λ_0 being a positive constant in both cases. In general, λ_k's do not necessarily have any physical meaning. They are usually chosen so that with a finite number of terms the Dirichlet series (1) can give a good approximation to f. In many recent papers (see Liu [6, 7] and the references cited therein) the relaxation times are simply chosen to be logarithmic equidistant spaced:

$$\lambda_k = \rho^k \lambda_0, \qquad k = 1, 2, \ldots, \tag{5}$$

where $\rho \neq 1$ and λ_0 are positive constant. By using the full Müntz Theorems in $C[0, 1]$ and in $L_p(0, 1)$ (Borwein and Erdélyi [2]) and a simple geometrical argument, we shall demonstrate that (5) is not necessarily a good choice from a mathematical point of view.

2. APPROXIMATION IN $C(R_+)$

First, we recall the definitions of absolutely monotonic functions and completely monotonic functions (see Widder [11]).

Definition 1. A function $f(x)$ is absolutely monotonic in the interval (a, b) if it has nonnegative derivatives of all orders there:

$$f^{(k)}(x) \geqslant 0, \qquad x \in (a, b), \quad k = 0, 1, 2, \ldots .$$

It is absolutely monotonic in $[a, b)$, ($(a, b]$ or $[a, b]$) if it is continuous there and is absolutely monotonic in (a, b).

Definition 2. A function $f(x)$ is completely monotonic in (a, b) ($[a, b)$ or $(a, b]$ or $[a, b]$) if $f(-x)$ is absolutely monotonic in $(-b, -a)$ ($(-b, -a]$ or $[-b, -a)$ or $[-b, -a]$).

An equivalent definition of absolutely monotonic functions due to Bernstein (see Widder [11]) that makes less continuity requirement is as follows.

Definition 3. A function $f(x)$ is absolutely monotonic in the interval $[a, b)$ if and only if

$$\Delta_h^m f(x) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + kh) \geqslant 0$$

for all nonnegative integers m and all $x \geqslant a$ and $h > 0$ such that $x + mh < b$.

It is obvious from Definition 1 that Dirichlet series of the form (1) are completely monotonic in $[0, \infty)$. The following result gives a new characterization of completely monotonic functions.

THEOREM 4. *A function, defined on* R_+, *can be approximated arbitrarily closely by Dirichlet series of the form* (1) *in supremum norm in* R_+ *if and only if it is completely monotonic in* R_+.

Proof. First, we consider the "only if" part. Suppose that a function f defined in R_+ can be approximated arbitrarily closely by a sequence $\{f_n\}_{n=1}^{\infty}$ of Dirichlet series of the form (1) in supremum norm. Let $g(t) = f(-t)$ and $g_n(t) = f_n(-t)$, $n = 1, 2, \ldots$, $t \in (-\infty, 0]$. For any arbitrary constant $L < 0$ and positive integer n, g_n is absolutely monotonic in $[L, 0)$. Hence,

$$\Delta_h^m g_n(t) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g_n(t+kh) \geqslant 0$$

for all nonnegative integers m and all $t \geqslant L$ and $h > 0$ such that $t + mh < 0$. Let $n \to \infty$ in the preceding inequality, we obtain

$$\Delta_h^m g(t) \geqslant 0$$

which implies that g is absolutely monotonic in $[L, 0)$. Since g is continuous at $t = 0$ and L is arbitrary, g is absolutely monotonic in $(-\infty, 0]$. Therefore, f is completely monotonic in $[0, \infty)$.

Next, we consider the "if" part. Suppose that $f(t)$ is completely monotonic in $[0, \infty)$. A theorem due to Bernstein (see Widder [11, p. 160]) states that

$$f(t) = \int_0^\infty \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau), \qquad t \in [0, \infty),$$

where β is bounded and non-decreasing and the integral converges for $t \in [0, \infty)$. For any $r > 0$, f can be rewritten as

$$f(t) = \int_0^r \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau) + \int_r^{1/r} \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau) + \int_{1/r}^\infty \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau)$$

where the first and the third integrals convergence to $\beta(0+) - \beta(0)$ and 0, respectively, as $r \to 0$, both uniformly with respect to $t \in \mathrm{R}_+$, because

$$\beta(0+) - \beta(0) \leqslant \int_0^r \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau) \leqslant \beta(r) - \beta(0),$$

$$0 \leqslant \int_{1/r}^\infty \mathrm{e}^{-t\tau}\, \mathrm{d}\beta(\tau) \leqslant \beta(\infty) - \beta(1/r).$$

Let $r=\tau_0<\tau_1<\cdots<\tau_m=1/r$ be a subdivision of the interval $[r, 1/r]$, and $\xi_i \in [\tau_i, \tau_{i+1}]$, $i=0, 1, \ldots, m-1$. Since

$$0<\mathrm{e}^{-t\tau_i}-\mathrm{e}^{-t\tau_{i+1}}<\frac{\tau_{i+1}-\tau_i}{\tau_i}\leqslant\frac{\tau_{i+1}-\tau_i}{r},$$

the sum

$$\sum_{i=0}^{m-1}\left(\beta(\tau_{i+1})-\beta(\tau_i)\right)\mathrm{e}^{-t\xi_i}$$

tends to

$$\int_r^{1/r}\mathrm{e}^{-t\tau}\,\mathrm{d}\beta(\tau)$$

uniformly with respect to $t\in\mathrm{R}_+$ as $\max_{0\leqslant i\leqslant m-1}(\tau_{i+1}-\tau_i)\to 0$. We conclude that f can be. approximated arbitrarily closely by Dirichlet series of the form

$$\beta(0+)-\beta(0)+\sum_{i=0}^{m-1}\left(\beta(\tau_{i+1})-\beta(\tau_i)\right)\mathrm{e}^{-t\xi_i}$$

in supremum norm. The coefficients in the preceding sum are nonnegative. This completes the proof. ∎

Going back to Joseph's question, we find that a smooth function, satisfying all or part of the assumptions listed in (2), cannot be approximated arbitrarily closely by Dirichlet series of the form (1) in terms of supremum norm, unless it is completely monotonic. A simple example is that of the function $f(t)=\mathrm{e}^{-t}-\mathrm{e}^{-2t}/4$.

3. APPROXIMATION IN $\mathrm{L}_p(0, \infty)$

To find a discrete spectral representation of the relaxation modulus in rheology the latest practice is to use nonnegative least square method (see Liu [6, 7] and the papers cited therein). Therefore, it is natural for us to investigate the problem of approximation by Dirichlet series with nonnegative coefficients in $\mathrm{L}^2(0, \infty)$, or in a slightly more general setting, in $\mathrm{L}_p(0, \infty)$, $p\in[1, \infty)$.

THEOREM 5. *Suppose that* $f \in L_p(0,\infty) \cap C(R_+)$, $1 \leqslant p < \infty$. *If* $f \in \overline{\text{Pspan}}\{e^{-\alpha t} \mid \alpha \in R_+ \mid L_p(R_+)\}$, *then it is a completely monotonic function in* R_+.

Proof. Consider the case $p \in (1,\infty)$. Let $q = p/(p-1)$ and

$$g(t) = e^{t/q} \int_t^{\infty} e^{-\tau/q} f(\tau)\, d\tau, \qquad b_k = \frac{\alpha_k}{\alpha_k + 1/q},$$

where a_k and α_k are arbitrary positive numbers. Since

$$g(t) - \sum_{k=1}^{n} b_k\, e^{-\alpha_k t} = e^{t/q} \int_t^{\infty} e^{\tau/q} \left[f(\tau) - \sum_{k=1}^{n} a_k\, e^{-\alpha_k \tau} \right] d\tau,$$

it follows from Hölder's inequality that

$$\left| g(t) - \sum_{k=1}^{n} b_k\, e^{-\alpha_k t} \right| \leqslant \left(\int_t^{\infty} \left| f(\tau) - \sum_{k=1}^{n} a_k\, e^{-\alpha_k \tau} \right|^p d\tau \right)^{1/p}.$$

If $f \in \overline{\text{Pspan}}\{e^{-\alpha t} \mid \alpha \in R_+ \mid L_p(R_+)\}$ then the preceding inequality implies that g can be approximated arbitrarily closely by Dirichlet series of the form (1) in $C(R_+)$. According to Theorem 4, g must be completely monotonic function in R_+. Since $f(t) = g(t)/q - g'(t)$, f is a completely monotonic function in R_+. The case of $p = 1$ can be dealt with in a similar fashion. ▌

We have not been able to determine whether a completely monotonic function in $L_p(0,\infty)$ can be approximated arbitrarily closely by Dirichlet series in L_p-norm.

4. ANALOGIES OF THE MÜNTZ THEOREMS

It has been stated in the Introduction that it is of practical importance to know whether it is alright to choose the exponents α_k in (1) from a given set such as (5) that is nowhere dense in R_+. The following theorem tells us that for a certain class of completely monotonic functions, there is an analogy of the Weierstrass Approximation Theorem.

THEOREM 6. *Suppose that* $f(\infty) = \lim_{t\to\infty} f(t) < \infty$ *and that*

$$g(x) := \begin{cases} f-(\log x), & x \in (0,1] \\ f(\infty), & x = 0 \end{cases}$$

is absolutely monotonic. Then $f \in \overline{\text{Pspan}}\{e^{-\alpha t} \mid \alpha = 0, 1, \ldots \mid C(R_+)\}$.

The preceding result follows trivially from Theorem 9b in Widder [11, p. 155] which states that such a function g call be approximated arbitrarily closely by polynomials with nonnegative coefficients.

THEOREM 7. *The completely monotonic function* $e^{-\beta t}$, $\beta > 0$ *and* $\beta \notin \{\alpha_k\}$, *is not in* $\overline{\text{Pspan}}\{e^{-\alpha t} \mid \alpha = \alpha_1, \alpha_2, \ldots \mid C(R_+)\}$ *if* $\beta < \liminf_{k\to\infty} \alpha_k$ *or* $\beta > \limsup_{k\to\infty} \alpha_k$.

This result can be proven by using a simple geometrical argument.

Finally, we present some results on the approximation by Dirichlet series without the restriction that tile coefficient are nonnegative. First, we cite the following two results that are alternative forms of the Müntz theorem in $C[0, 1]$ and in $L_p(0, \infty)$ due to Borwein and Erdélyi [2].

MÜNTZ THEOREM IN $C_R(R_+)$. *Let* $\{\alpha_k\}$ *be a infinite sequence of distinct positive numbers. Then* $\text{Span}\{1, e^{-\alpha_1 t}, e^{-\alpha_2 t}, \ldots\}$ *is dense in*

$$C_R(R_+) := \{y \in C(R_+) \mid y(\infty) = \lim_{t\to\infty} y(t) \text{ exists}\}$$

if and only if

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{1+\alpha_k^2} = \infty. \tag{6}$$

holds.

MÜNTZ THEOREM IN $L_p(0, \infty)$. *Let* $\{\alpha_k\}$ *be an infinite sequence of distinct positive numbers. Then* $\text{Span}\{e^{-\alpha_1 t}, e^{-\alpha_2 t}, \ldots\}$ *is dense in* $L_p(0, \infty)$, $1 \leqslant p < \infty$, *if and only if (6) holds.*

The following two results reveal the weakness in choosing a fixed $\Delta = \{1/\lambda_k\}_{k=1}^{\infty}$ with λ_k being of the form (3), (4) or (5).

THEOREM 8. *The completely monotonic function* $e^{-\beta t}$, $\beta > 0$ *and* $\beta \notin \{\alpha_k\}$, *is not in* $\overline{\text{Span}}\{e^{-\alpha t} \mid \alpha = \alpha_1, \alpha_2, \ldots, \mid C(R_+)\}$ *if*

$$\sum_{k=1}^{\infty} 1/\alpha_k < \infty \tag{7}$$

or

$$\eta := \sum_{k=1}^{\infty} \alpha_k < \infty, \qquad \beta > 11(1+\eta) \tag{8}$$

holds.

Proof. First, we consider the case where (7) holds. Let

$$d_n := \left\| e^{-\beta t} - \sum_{k=1}^{n} a_k^* e^{-\alpha_k t} \right\|_\infty = \min_{a_k} \left\| e^{-\beta t} - \sum_{k=1}^{n} a_k e^{-\alpha_k t} \right\|_\infty .$$

We have

$$\left\| e^{-(\beta+1/2)\,t} - \sum_{k=1}^{n} a_k^* e^{-(\alpha_k+1/2)\,t} \right\| \leqslant d_n \left(\int_0^\infty e^{-t}\, dt \right)^{1/2} = d_n$$

and

$$\left\| e^{-(\beta+1/2)\,t} - \sum_{k=1}^{n} a_k^* e^{-(\alpha_k+1/2)\,t} \right\|_2 \geqslant \min_{a_k} \left\| e^{-(\beta+1/2)\,t} - \sum_{k=1}^{n} a_k e^{-(\alpha_k+1/2)\,t} \right\|_2$$
$$= \frac{1}{\sqrt{2\beta+1}} \prod_{k=1}^{n} \frac{|\alpha_k - \beta|}{\alpha_k + \beta + 1}$$

the last identity follows from Lemma 11.3.2 in [3]. If $\beta \notin \{\alpha_k\}$ and $\sum_{k=1}^{\infty} 1/\alpha_k < \infty$ then

$$d_n > \frac{1}{\sqrt{2\beta+1}} \prod_{k=1}^{\infty} \frac{|\alpha_k - \beta|}{\alpha_k + \beta + 1} > 0$$

which means that the function $e^{-\beta t}$ can not be approximated arbitrarily closely by functions from $\mathrm{Span}\{e^{-\alpha t} \mid \alpha = \alpha_1, \alpha_2, ..., \mid C(R_+)\}$.

For the case $\eta := \sum_{k=1}^{\infty} \alpha_k < \infty$, we need the inequality

$$\|p'(t)\|_\infty \leqslant 11(1+\eta)\, \|p(t)\|_\infty \qquad \forall p \in \mathrm{Span}\{e^{-\alpha_1 t}, e^{-\alpha_2 t}, ...\}, \tag{9}$$

which is equivalent to Newman's inequality [8]. Using the Mean Value Theorem and the inequality (9), it can be shown by contradiction that when $\beta > 11(1+\eta)$ the function $e^{-\beta t}$ cannot, be approximated arbitrarily closely by functions from $\mathrm{Span}\{e^{-\alpha t} \mid \alpha = \alpha_1, \alpha_2, ... \mid C(R_+)\}$. ▮

THEOREM 9. *The completely monotonic function* $e^{-\beta t}$, $\beta > 0$ *and* $\beta \notin \{\alpha_k\}$, *is not in* $\overline{\mathrm{Span}}\{e^{-\alpha t} \mid \alpha = \alpha_1, \alpha_2, ... \mid L_2(R_+)\}$ *unless* (6) *is satisfied.*

Proof. It follows from Lemma 11.3.2 in [3] that

$$\min_{a_k} \left\| e^{-\beta t} - \sum_{k=1}^{n} a_k\, e^{-\alpha_k t} \right\|_2 = \frac{1}{\sqrt{2\beta}} \prod_{k=1}^{n} \frac{|\alpha_k - \beta|}{\alpha_k + \beta}$$

which has a positive lower bound if (6) is not satisfied. ▮

There are still many questions left unanswered. It is still unknown whether completely monotonic functions in $L_p(R_+)$ can be approximated arbitrarily closed by Dirichlet series with nonnegative coefficients in L_p-norm. Most akin to the application in rheology, we do not know whether there are analogies of the Müntz Theorems for completely monotonic functions. A conjecture we would like to make is that a necessary and sufficient condition for having $\text{Pspan}\{e^{-\alpha t} \mid \alpha = 0, \alpha_1, \alpha_2, \ldots \mid C(R_+)\}$ dense in the set of completely monotonic functions in R_+ is that the set of distinct positive numbers $\{\alpha_1, \alpha_2, \ldots\}$ satisfies $\liminf_{k\to\infty} \alpha_k = 0$, $\limsup_{k\to\infty} \alpha_k = \infty$ and the identity (6).

ACKNOWLEDGMENTS

The author thanks Tamás Erdélyi (Texas A & M University) for providing details of the proof of the Müntz theorem in C[0, 1] in the paper [2], Arieh Iserles (Cambridge University) for many invaluable discussions on Dirichlet series, and an anonymous referee for spotting a number of typos and misprints in an earlier version of this paper.

REFERENCES

1. B. J. C. Baxter and A. Iserles, On approximation by exponentials, *Ann. Numer. Math.* **4** (1997), 39–54.
2. P. Borwein and T. Erdélyi, Full Müntz theorem in C[0, 1] and $L_1[0, 1]$, *J. London Math. Soc. (2)* **54** (1996), 102–110.
3. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
4. M. Doi and S. F. Edwards, "The Theory of Polymer Dynamics," Oxford Univ. Press, London, 1986.
5. D. D. Joseph, "Fluid Dynamics of Viscoelastic Liquids," Springer-Verlag, New York/Berlin, 1990.
6. Y. Liu, Calculation of discrete relaxation modulus and creep compliance, *Rheologica Acta* **38** (1999), 357–364.
7. Y. Liu, A direct method for obtaining discrete relaxation spectra from creep data, *Rheologica Acta* **40** (2001), 256–260.
8. D. J. Newman, Derivative bounds for Müntz polynomials, *J. Approx. Theory* **18** (1976), 360–362.
9. V. Operstein, Full Müntz theorem in $L_p[0, 1]$, *J. Approx. Theory* **85** (1996), 233–235.
10. N. W. Tschoegl, "The Phenomenological Theory of Linear Viscoelastic Behaviour," Springer-Verlag, New York/Berlin, 1989.
11. D. V. Widder, "The Laplace Transform," Princeton Univ. Press, Princeton, NJ , 1946.